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Transition Radiation X-Ray Laser Based on Stimulated Processes at the Boundary between Two Dielectric Media

Kenneth E. Okoye and Hironari Yamada

Abstract— **This paper analyzes a model of a transition radiation laser based on stimulated emission induced by relativistic electrons crossing the boundary between two media of different dielectric properties. Interaction between the incident radiation and the electrons in this boundary region is taken into account. Phenomenological quantum electrodynamics is applied to derive analytical expressions for stimulated emission and absorption probabilities. Analogs of Einstein's coefficients for the transition processes have also been derived and discussed. It is shown that stimulated emission is greater than absorption. The gain is then calculated.**

Index Terms— **Absorption, laser, gain, stimulated emission , transition radiation.**

I. Introduction

The operation of classical laser (CL) is based on the occurrence of population inversion achieved by pumping of laser active medium and the attendant stimulated emission and absorption of radiation as electrons transit between energy levels in the laser medium in the presence of an incident radiation as proposed by Einstein. The laser wavelength of CL is determined by the nature of the laser medium. CL is a very compact system but it is difficult to generate short wavelength because the lifetime for short wavelength is too short to achieve population inversion.

In free electron laser (FEL), a beam of electrons is made to transverse a periodic undulator magnetic field which induces transverse oscillations on the electrons. The undulating electron beam interacts with incident electromagnetic wave (EM) leading to micro bunching of the electron beam and the amplification of the radiation. The FEL wavelength is defined by the period of the undulator. It is thus artificially fixed. Phase matching between electron beam and EM wave is necessary for the operation of FEL. The generation of short wave length by FEL however requires very long and stable system such as 1 km electron LINAC and 100m long undulator [1]. A major technological breakthrough will be the development of a compact system for generation of short laser wavelength.

The mechanisms to spontaneously generate short wavelength radiation are historically well known in literature. They include Bremsstrahlung [2], which is generated by the collision of solid target and relativistic electrons, and Transition radiation [3], which is emitted when an electron crosses the boundary between two media of different dielectric constants. Resonance transition radiation (RTR) using a periodic multi layer foil or stack foils has been reported by many groups [4]. Use of micro bunched beam is also proposed to generate coherent interaction [5]- [8], but any gain is yet to be reported.

A novel laser scheme proposed by Yamada [9] combines FEL mechanism, Einstein's forced radiation mechanism and one out of the following: Bremsstrahlung, Parametric radiation or Transition radiation. The mechanism which selects the wavelength is introduced in this novel scheme similar to SASE-FEL. One of the periodic interactions of the radiation scheme is shown in Fig. 1. Spontaneous radiation is generated at the first stage by thin targets (not shown). This radiation is then monochromatized by a crystal. When the target itself is made of a thin crystal, monochromatic radiation is generated in the direction of the specific angle θ. This radiation is called parametric radiation. Transition radiation mechanism is useful to generate EUV and soft X-ray radiations with electrons of energy around 10 MeV. When the monochromatic radiation and electron beam merge at some angle, we expect a positive interaction. The electric field toward the electron velocity appears as E∗cos(θ), thus the interaction takes place. The magnet C is introduced to adjust the timing between electrons and EM field.

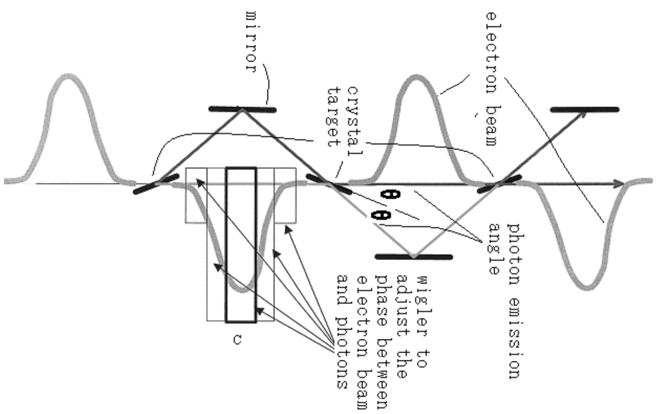

Fig. 1. Novel laser scheme combining FEL and Einstein's stimulated radiation schemes. In this diagram the parametric X-ray is generated by the target, but Bremsstrahlung or transition radiation is also useful when monochrometer is introduced instead of mirror.

In this paper we will present a quantum electrodynamical treatment of stimulated emission and absorption of radiation by a beam of relativistic electrons crossing the interface between two dielectric media as a result of their interaction with an external electromagnetic radiation at the interface. The amplification gain by the single interface is calculated, and a model of transition radiation laser based on these stimulated processes occurring at the dielectric interface is presented. We are concerned with the system where multiple foils are periodically aligned. Each foil has two boundaries, then the distance between two boundaries satisfy the resonance condition, and the distance between any two foils satisfies the matching condition as well. Our laser concept is fundamentally different from RTR laser analyzed in [8]. The RTR laser is similar to conventional FEL, thus the micro bunching process of electron beam is taken into accounts. The laser we analyzed here is in some way similar to conventional lasers except that free relativistic electrons are involved, but the micro bunching process is unnecessary.

We take a quantum mechanical approach to justify that the classical concept of the stimulated emission is applicable to the transition radiation. Then we will find that the classical approach is useful to study the gain of the laser.

S. Datta and A.E. Kaplan have provided a quantum mechanical treatment of stimulated resonant transition but as a special case of "Cerenkov" emission into higher-order spatial harmonics [10]. Our approach in calculating amplification gain at each interface follows closely the analysis of spontaneous emission of transition radiation based on phenomenological quantum electrodynamics by Garibyan [11]. A somewhat similar analysis has been provided by Zaretskii et. al. but they analyzed stimulated one-photon transition processes involving non relativistic electrons[12]. Here we are concerned with the analyses of stimulated one-photon transition processes involving relativistic electrons.

The operation of the laser requires successive overlap of the electron bunches and the radiation at the boundaries of the dielectric media.

Section III presents analyses of stimulated processes occurring at the interface between two dielectric media, and the calculation of gain. Section IV discusses the analogs of Einstein coefficients associated with the transition processes. Section V provides an analysis of a model of transition radiation laser based on stimulated processes considered together with the proposition of a coherent gain.

II. SPONTANEOUS EMISSION

Calculation of spontaneous emission rate of transition radiation from relativistic electrons based on phenomenological quantum electrodynamics has been done many years ago by Garibyan[11]. The differential spectral yield is:

$$\frac{dN_o}{d\Omega d\omega} = \frac{e^2\omega}{16\pi^3}\,\frac{E_2E_1 - P_1P_{2z}\cos^2\theta - M^2}{E_1P_{2z}}\left[\frac{1}{P_{1z}-P_{2z}-K_{1z}} - \frac{1}{P_{1z}-P_{2z}-K_{2z}}\right]^2 \quad (1)$$

where N_o is the photon number, E_1 is the incident energy of the electron, E_2 is the energy of the electron after emission of a photon, P_1 is the incident electron momentum, P_2 is the electron momentum after emission of a photon, K is the momentum of the photon, M is the electron mass, θ is the angle of emission as shown in Fig 2.

III. STIMULATED TRANSITION PROCESSES

A. Stimulated Transition Radiation Emission

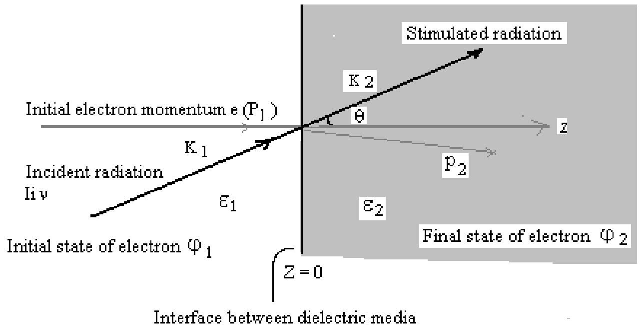

Fig. 2. Stimulated transition radiation at the interface of two media: the refractive index changes from ε_1 to ε_2 at the interface

We will calculate the stimulated emission and absorption rate of transition radiation following Ref. [11]. With reference to Fig. 2, we consider a single electron of definite momentum $\mathbf{P_i}$ crossing the boundary between two media with dielectric permittivities ε_1 and ε_2 . Also incident on the dielectric interface is a monochromatic radiation I_{iv} of frequency, ω. We assume the position of the dielectric boundary to be at Z=0

First we shall calculate the probability of stimulated emission by one electron.

We employ natural units such that C = ħ = 1

Where C is the velocity of light, ħ is the modified Planck constant. The initial state $|I\rangle$ of the system has N photons in the state $|N_{k,\omega}\rangle$ and the electron in the momentum state

$$|P_i\rangle = \mathbf{C}^+(P_1)\;|\mathbf{0}\rangle$$

$$|I\rangle = |P_i\rangle\,|N_{k,\omega}\rangle \quad (2)$$

We consider transition to a final state $|F\rangle$ of the system with the electron in the state $|\mathbf{P_2}\rangle = \mathbf{C}^+(P_2)\;|\mathbf{0}\rangle$ and N+1 photons in the state $|N+1_{k,\omega}\rangle$.

$$|F\rangle = \;|P_2\rangle\,|N+1_{k,\omega}\rangle \quad (3)$$

The initial state thus represents an electron of momentum P_1 and energy E_1 in the spinor state $U(P_1)$, and N photons of energy ω while the final state has electron of momentum P_2 and E_2 in the spinor state $U(P_2)$.

The effect under consideration is a first order process.

The matrix element for the process is represented by:

$$M_{fi} = \langle F|S|I\rangle$$

$$S = e\int d^4x\, N(\overline{\varphi}(x)\gamma^\mu A_\mu(x)\varphi(x)) \quad (4)$$

Where γ^μ are the gamma matrices.

$$\varphi(x) = \sqrt{\frac{M}{EV}}\,u(p)\{\mathbf{C}^+(p)\exp(-ipx) + C(p)(\exp(ipx)\}$$

where $C^+(p)$ and $C(p)$ are creation and absorption operators for electrons respectively and u(p) is the positive-energy solution of Dirac equations with four momentum P^μ. We limit ourselves to relativistic particles. For X-ray radiation, we assume that the dielectric constant ε is expressed in terms of the plasma frequencies ω_p, $\varepsilon = 1 - \omega_p^2/\omega^2$ where $\omega >> \omega_p$ and thus that the radiation from medium 1 ($z < 0$) propagates through the interface without reflection or refraction [13]

$A_\mu(x)$ for the problem is then given by [11]:

$$A_\mu(x) = \sum_{m,\omega} \frac{1}{\sqrt{2V\omega}} \begin{Bmatrix} \theta(-z)\frac{e_\mu{}^m}{\sqrt{\varepsilon_1}}\left(ae^{ik_1x} + a^+e^{-ik_1x}\right) \\ +\theta(z)\frac{e_\mu{}^m}{\sqrt{\varepsilon_2}}\left(ae^{ik_2x} + a^+e^{-ik_2x}\right) \end{Bmatrix} \quad (5)$$

where $\theta(z) = 1$ for positive values of z
$\theta(z) = 0$ for negative values of z

For purpose of quantization, a^+ and a in (5) represent creation and absorption operator respectively. Note that in this case of stimulated emission only the terms containing creation operators are applicable.

The probability per unit time for stimulated emission is:

$$\frac{|M_{fi}|^2}{T} = (N+1)\frac{e^2}{V^3}\frac{M^2}{E_2E_1}\frac{(2\pi)^3}{2\omega} S\,\delta(P_{1x} - P_{2x} - K_{1x}) \times \delta(P_{1y} - P_{2y} - K_{1y})\delta(E_1 - E_2 - \omega) \times \left|\bar{U}(P_2)\left(\frac{\hat{e}^m}{\sqrt{\varepsilon_1}(P_{1z}-P_{2z}-K_{1z})} - \frac{\hat{e}^m}{\sqrt{\varepsilon_2}(P_{1z}-P_{2z}-K_{2z})}\right)U(P_1)\right|^2 \quad (6)$$

where we used $(2\pi)\delta(P) = \int e^{ipx}\, d^4x$, $\hat{e}^m = \gamma^\mu e_\mu{}^m$.

S designates the cross section of the target, and T is the time of interaction. From (6) it is seen that the probability of stimulated emission of radiation is dependent on the number (N+1) of photons present which is characteristic of stimulated emission. The case N =0 corresponds to spontaneous emission. Equation 6 gives the probability rate of stimulated emission of transition radiation photon of a definite momentum from an electron.

For total probability, we integrate over all possible P_2 and average over initial electron spin and sum over final electron spin states. We have:

$$W_e = \frac{N+1}{V}\frac{e^2}{2\omega}\frac{E_2E_1 - P_1P_{2z}\cos^2\theta - M^2}{E_1P_{2z}} \times \left(\frac{1}{\sqrt{\varepsilon_1}(P_{1z}-P_{2z}-K_{1z})} - \frac{1}{\sqrt{\varepsilon_2}(P_{1z}-P_{2z}-K_{2z})}\right)^2$$

$$= \frac{N+1}{V}\frac{e^2}{2\omega}\frac{E_2E_1 - P_1P_{2z}\cos^2\theta - M^2}{E_1P_{2z}} \times \left(\frac{1}{\sqrt{\varepsilon_1}(P_{1z}-P_{2z}-\sqrt{\varepsilon_1}\omega\cos\theta)} - \frac{1}{\sqrt{\varepsilon_2}(P_{1z}-P_{2z}-\sqrt{\varepsilon_2}\omega\cos\theta)}\right)^2 \quad (7)$$

where we note that:

$$K_{2z} = \sqrt{\varepsilon_2}\,\omega\cos\theta \;;\; K_{1z} = \sqrt{\varepsilon_1}\,\omega\cos\theta$$
$$E_2 = E_1 - \omega,\; P_{2t}{}^2 = \varepsilon_2\omega^2\sin^2\theta \;;$$
$$P_{2t}{}^2 + P_{2z}{}^2 + M^2 = E_2{}^2$$

where P_{2z} is the z-component of the electron momentum after emission of a photon. The $\frac{1}{(P_{1z}-P_{2z}-\sqrt{\varepsilon}\omega\cos\theta)}$ is defined as the formation length[13].

The stimulated emission rate F_{iv} from a current density J when the interface is illuminated by the photon flux, I_{iv}, is thus given by:

$$F_{iv} = I_{iv}\frac{J}{e}\frac{e^2}{2\omega}\frac{E_2E_1 - P_1P_{2z}\cos^2\theta - M^2}{E_1P_{2z}} \times \left(\frac{1}{\sqrt{\varepsilon_1}(P_{1z}-P_{2z}-\sqrt{\varepsilon_1}\omega\cos\theta)} - \frac{1}{\sqrt{\varepsilon_2}(P_{1z}-P_{2z}-\sqrt{\varepsilon_2}\omega\cos\theta)}\right)^2 \quad (8)$$

B. Stimulated Transition Radiation Absorption

In addition to stimulated emission, we also have stimulated absorption of radiation at the boundary between two dielectric material as a result of the interaction of radiation and electrons crossing the interface. The analysis of stimulated absorption is similar to that of stimulated emission except that the final state $|F\rangle$ of the system has an electron in the state $|\mathbf{P_{2a}}\rangle = \mathbf{C}^+(P_{2a})\,|\mathbf{0}\rangle$ and N−1 photons in the state $|N-1_{k,\omega}\rangle$.

$$|F\rangle = |P_{2a}\rangle|N-1_{k,\omega}\rangle \quad (9)$$

Thus the final state has electron of momentum P_{2a} and energy E_{2a} in the spinor state $U(P_{2a})$,
The subscript 'a' above implies the case of absorption. We also note that in the case of stimulated absorption only the terms in (5) containing absorption operators are applicable. Following exactly the same procedure as in the case of stimulated emission, we have for the total absorption probabilityW_a:

$$W_a = \frac{N}{V}\frac{e^2}{2\omega}\frac{E_{2a}E_1 - P_1P_{2za}\cos^2\theta - M^2}{E_1P_{2za}} \times \left(\frac{1}{\sqrt{\varepsilon_1}(P_{1z}-P_{2za}+\sqrt{\varepsilon_1}\omega\cos\theta)} - \frac{1}{\sqrt{\varepsilon_2}(P_{1z}-P_{2za}+\sqrt{\varepsilon_2}\omega\cos\theta)}\right)^2 \quad (10)$$

where we note that:

$$E_2 = E_1 + \omega,\; P_{2t}{}^2 = \varepsilon_2\omega^2\sin^2\theta \;;$$
$$P_{2t}{}^2 + P_{2za}{}^2 + M^2 = E_2{}^2$$

where P_{2za} is the z-component of the electron momentum after absorption of a photon. The $\frac{1}{(P_{1z}-P_{2za}+\sqrt{\varepsilon}\omega\cos\theta)}$ is defined as the absorption length.

The stimulated absorption rate μI_{iv} by a current density J when the interface is illuminated by the photon flux, I_{iv}, is thus given by:

$$\mu I_{iv} = I_{iv}\frac{J}{e}\frac{e^2}{2\omega}\frac{E_2E_1 - P_1P_{2za}\cos^2\theta - M^2}{E_1P_{2za}} \times \left(\frac{1}{\sqrt{\varepsilon_1}(P_{1z}-P_{2za}+\sqrt{\varepsilon_1}\omega\cos\theta)} - \frac{1}{\sqrt{\varepsilon_2}(P_{1z}-P_{2za}+\sqrt{\varepsilon_2}\omega\cos\theta)}\right)^2 \quad (11)$$

C. Gain Coefficient per Interface g

The net change of flux through the interface ΔI_{iv} is:

$$\Delta I_{iv} = F_{iv} - \mu I_{iv}$$

The gain coefficient g per interface is given by:

$$g = \frac{\Delta I_{iv}}{I_{iv}} \quad (12)$$

From equations 8, 11 &12, the gain g simplifies to:

$$g = \frac{J}{e}\frac{e^2}{2\omega}\frac{E_2E_1 - P_1P_{2z}\cos^2\theta - M^2}{E_1P_{2z}} \times \left(\frac{1}{\sqrt{\varepsilon_1}(P_{1z} - P_{2z} - \sqrt{\varepsilon_1}\omega\cos\theta)} - \frac{1}{\sqrt{\varepsilon_2}(P_{1z} - P_{2z} - \sqrt{\varepsilon_2}\omega\cos\theta)}\right)^2 \times \left(\frac{1}{P_{2z}} - \frac{1}{P_{2za}}\right) \quad (13)$$

IV. ANALOGS OF EINSTEIN`S COEFFICIENTS A AND B

The transition probability per unit time is given by (6). There are two contributions to this emission rate. The part linear in N gives a rate proportional to the number N of photons present initially and corresponds to stimulated emission. The other contribution independent of N corresponds to spontaneous radiation and occurs in the absence of any radiation, when the radiation field is initially in its vacuum state. We now derive the analogs of Einstein's coefficients A and B which are respectively related to spontaneous emission probability rate and stimulated emission probability rate. We note that there are differences between the atomic case for which these coefficients were originally derived by Einstein and the case of transition radiations which we treat here. These differences given in [14] are: (1) contrary to case of transition radiation here treated, the transition energy levels of the atomic radiators are independent of radiation frequency and emission angle. (2) The radiation frequency is independent of emission direction. (3) In Einstein formulation both the spontaneous and stimulated emission are discussed in terms of a continuum of radiation modes. In view of these differences some care should be taken in the definition of emission rate parameters.

A Einstein's Coefficient A

The Einstein's coefficient A is equivalent to the total spontaneous emission probability rate for a photon with energy ω. Because of the frequency dependence on the direction of spontaneous emission of transition radiation we define the total spontaneous emission probability rate to correspond to the total probability of spontaneous emission into the differential solid angle and frequency segment $\triangle\Omega$, $\triangle\omega$ which is equivalent to a linewidth for emission around a certain emission frequency center ω_o and emission direction. Thus A coefficient is calculated by integrating the spontaneous part of (6) over all possible P_2 and averaging over initial electron spin states and summing over final electron spin states and multiplying by the number of radiation modes within the solid angle $\triangle\Omega$ and frequency band width $\triangle\omega$.

Noting that $\Delta^3K = \omega^2\triangle\Omega\triangle\omega$, we have:

$$\frac{1}{\tau} = A = \frac{e^2\omega}{16\pi^3L}\frac{E_2E_1 - P_1P_{2z}\cos^2\theta - M^2}{E_1P_{2z}} \times \left(\frac{1}{\sqrt{\varepsilon_1}(P_{1z} - P_{2z} - \sqrt{\varepsilon_1}\omega\cos\theta)} - \frac{1}{\sqrt{\varepsilon_2}(P_{1z} - P_{2z} - \sqrt{\varepsilon_2}\omega\cos\theta)}\right)^2 \Delta\,\Omega\,\Delta\,\omega \quad (14)$$

Equation (14) gives an analog of Einstein coefficient A for the transition radiation process.

B. Einstein's Coefficient B for Stimulated Emission

Einstein's Coefficient B for stimulated emission is derived from stimulated emission probability rate. The B coefficient is defined by:

$$W_e = B \times U(\omega), \quad (15)$$

where U(ω) represents the energy density of the radiation modes. U(ω) is given by [14]:

$$U(\omega) = \omega N\rho(\omega) \quad (16)$$

$$\rho(\omega) = \frac{\omega^2}{\pi^2} \quad (17)$$

In view of stimulated emission into a continuum of radiation modes, we multiply stimulated emission part of (6) by the number of radiation modes within the solid angle $\triangle\Omega$ and frequency band width $\triangle\omega$ and integrate over all possible P_2 and averaging over initial electron spin states and summing over final electron spin states. We have:

$$W_e = N\frac{e^2\omega}{16\pi^3L}\frac{E_2E_1 - P_1P_{2z}\cos^2\theta - M^2}{E_1P_{2z}} \times \left(\frac{1}{\sqrt{\varepsilon_1}(P_{1z} - P_{2z} - \sqrt{\varepsilon_1}\omega\cos\theta)} - \frac{1}{\sqrt{\varepsilon_2}(P_{1z} - P_{2z} - \sqrt{\varepsilon_2}\omega\cos\theta)}\right)^2 \Delta\,\Omega\,\Delta\,\omega \quad (18)$$

From (15),(16) and (17), we have:

$$B = \frac{\pi^2}{\omega^3}\frac{e^2\omega}{16\pi^3L}\frac{E_2E_1 - P_1P_{2z}\cos^2\theta - M^2}{E_1P_{2z}} \times \left(\frac{1}{\sqrt{\varepsilon_1}(P_{1z} - P_{2z} - \sqrt{\varepsilon_1}\omega\cos\theta)} - \frac{1}{\sqrt{\varepsilon_2}(P_{1z} - P_{2z} - \sqrt{\varepsilon_2}\omega\cos\theta)}\right)^2 \Delta\,\Omega\,\Delta\,\omega \quad (19)$$

Equation 19 gives an analog of Einstein coefficient B for the stimulated transition radiation process.

The ratio of Coefficients A for spontaneous emission and B for stimulated emission is:

$$\frac{A}{B} = \frac{\omega^3}{\pi^2} \quad (20)$$

Thus this satisfies Einstein relations.

C. Einstein's Coefficient B for Absorption

The analog of the third Einstein coefficient B for absorption can be derived following the procedure for deriving Einstein B coefficient for stimulated emission. It is derived from stimulated absorption probability rate. The B coefficient is defined by:

$$W_a = B \times U(\omega). \quad (21)$$

Starting from (10) and following similar procedure as in stimulated emission, we have:

$$B = \frac{\pi^2}{\omega^3}\frac{e^2\omega}{16\pi^3 L}\frac{E_{2a}E_1 - P_1 P_{2za}\cos^2\theta - M^2}{E_1 P_{2za}} \times \left(\frac{1}{\sqrt{\varepsilon_1}(P_{1z} - P_{2za} + \sqrt{\varepsilon_1}\omega\cos\theta)} - \frac{1}{\sqrt{\varepsilon_2}(P_{1z} - P_{2za} + \sqrt{\varepsilon_2}\omega\cos\theta)}\right)^2 \Delta\,\Omega\,\Delta\,\omega \quad (22)$$

Equation 22 gives an analog of Einstein coefficient B for the stimulated absorption process.

V. LASER DESIGN

The analysis carried out in section III above relates basically to the amplification of radiation at the boundary region of two dielectric media as a result of its interaction with relativistic electrons crossing the interface. In the application of the results of the foregoing analysis in the design of a laser, it is necessary that the laser fields should be made to grow from spontaneous emission or a suitable radiation source. The laser configuration should also operate at a chosen frequency.
Spontaneously emitted transition radiation diverges from the electrons and thus does not easily make for stimulated emission. Stimulated emission requires that electron beams and radiation overlap. One way to achieve this is to wiggle or spiral the electrons such that the radiation adds along the axis of the wiggles as is done in the case of the PDFEL and RTR lasers [15]. However a suitable method in this case is to use a suitably positioned mirror or crystal monochrometer to reflect the radiation so as to merge them with the electrons at an angle at the interface. This is employed in photon storage ring [16]. Fig. 3 illustrates the operation of this laser.

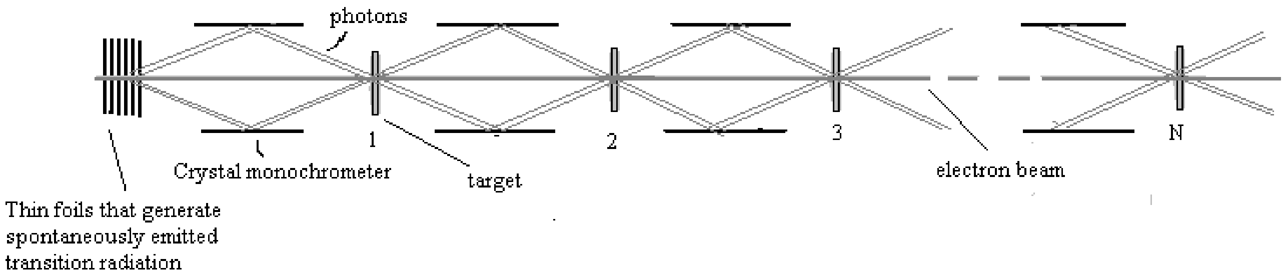

Fig. 3. The laser configuration.

In the laser design as shown in figure 3, spontaneous transition radiations is emitted from a number of thin foil stack and monochromatized by the crystal monochrometer and then concentrated to merge and interact with the electron beam at the boundaries of the dielectric media in the laser medium. The electron beam is composed of electron bunches.

A Gain coefficient for a single foil

We considered a single interface in our analysis in section III. Thin foils are, however, employed in the laser design. The foil has two interfaces. We now consider the amplification of radiation crossing a single foil of permittivity ε_2 and thickness L placed in a medium of permittivity ε_1 as shown in fig. 4.

The free electromagnetic potential $A_\mu(x)$ through the foil takes the form:

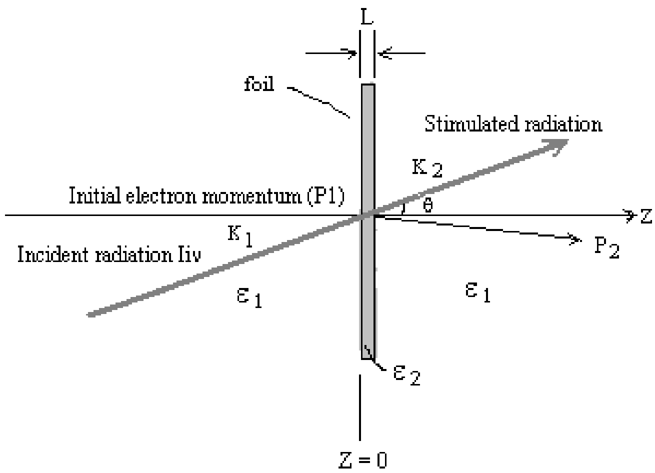

Fig. 4. Stimulated transition radiation from a thin foil having two interfaces

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$$A_\mu(x) = \sum_{m,\omega} \frac{1}{\sqrt{2V\omega}} \begin{Bmatrix} \theta(-z)\frac{e_\mu{}^m}{\sqrt{\varepsilon_1}}\left(ae^{ik_1x} + a^+e^{-ik_1x}\right) \\ +\theta(z)\frac{e_\mu{}^m}{\sqrt{\varepsilon_2}}\left(ae^{ik_2x} + a^+e^{-ik_2x}\right) \\ -\theta(z-L)\frac{e_\mu{}^m}{\sqrt{\varepsilon_2}}\left(ae^{ik_2x} + a^+e^{-ik_2x}\right) \\ +\theta(z-L)\frac{e_\mu{}^m}{\sqrt{\varepsilon_1}}\left(ae^{ik_1x} + a^+e^{-ik_1x}\right) \end{Bmatrix} \quad (23)$$

where $\theta(z-L) = 1$ for $z > L$
$\theta(z-L) = 0$ for $z < L$

By substituting (23) into (4) and following similar procedure as in section III we have:

$$G_f = \frac{J}{e}\,\frac{e^2}{2\omega}\,\frac{E_2E_1 - P_1P_{2z}\cos^2\theta - M^2}{E_1P_{2z}} \times \left(\frac{1}{\sqrt{\varepsilon_1}(P_{1z}-P_{2z}-\sqrt{\varepsilon_1}\omega\cos\theta)} - \frac{1}{\sqrt{\varepsilon_2}(P_{1z}-P_{2z}-\sqrt{\varepsilon_2}\omega\cos\theta)}\right)^2 \times \left(\frac{1}{P_{2z}} - \frac{1}{P_{2za}}\right)\left|1 - \exp i\{(P_{1z} - P_{2z} - \sqrt{\varepsilon_2}\omega\cos\theta)L\}\right|^2$$

$$= g\left|1 - \exp i\{(P_{1z} - P_{2z} - \sqrt{\varepsilon_2}\omega\cos\theta)L\}\right|^2 \quad (24)$$

where G_f is the gain coefficient for a single foil; and g is the gain coefficient per interface given by (13).

Using $1 - \cos 2\theta = 2\sin^2\theta$, we have:

$$G_f = 4g\sin^2\left\{\frac{(P_{1z}-P_{2z}-\sqrt{\varepsilon_2}\omega\cos\theta)L}{2}\right\} \quad (25)$$

We see that the gain for a single foil depends on the phase $\frac{(P_{1z}-P_{2z}-\sqrt{\varepsilon_2}\omega\cos\theta)L}{2}$. The maximum gain per foil, G_{fmax} is thus:

$$G_{fmax} = 4g \quad (26)$$

From (25), we see that this is achieved when:

$$\frac{(P_{1z}-P_{2z}-\sqrt{\varepsilon_2}\omega\cos\theta)L}{2} = (M - \tfrac{1}{2})\pi \quad (27)$$

Equation 27 corresponds exactly to the resonance condition for the emission of resonance transition radiation [5, 15, 17] where the emission from the foil is four times that from each interface. The minimum gain per foil, G_{fmin} is given by [17]:

$$G_{fmax} = 2g \quad (28)$$

So is twice the gain for each interface.

B Matching Condition

The operation of the laser requires successive overlap of the electron bunches and the radiation at the boundaries of the dielectric media. This is illustrated in Fig. 5.

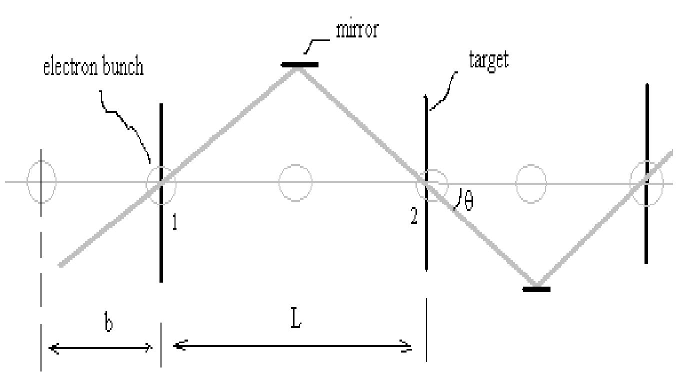

Fig. 5 Matching condition

This overlap requires a matching condition which we now derive. The photon moves from point 1 to point 2 in time $\triangle T_1$ given by:

$$\triangle T_1 = \frac{L}{C\cos\theta} \quad (29)$$

While the electron moves from point 1 to point 2 in time $\triangle T_2$ given by:

$$\triangle T_2 = \frac{L+nb}{V_e} \quad (30)$$

Where n is an integer and b is the distance between the electron bunches. The matching condition requires

$\triangle T_1 = \triangle T_2$:

$$\frac{L}{C\cos\theta} = \frac{L+nb}{V_e} \quad (31)$$

The matching condition becomes:

$$\frac{1}{\cos\theta} = \frac{1}{\beta}\left\{1 - n\frac{b}{L}\right\} \quad (32)$$

Thus we can choose n and b for a desired angle.

C. Laser Gain.

The gain given by (25) above is the gain coefficient representing the gain for a single foil. The gain for the entire laser configuration is derived following the diagram below.

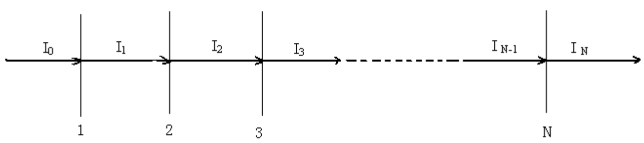

Fig. 6. Successive amplification after each foil

The numbers in the fig. 6 above represent the foil. We represent the gain at each foil by G_f. We assume the incident radiation to be I_0. Then the radiation after the foil 1 is I_1

$$I_1 = I_0 + gI_o = I_o(1 + G_f) \quad (33)$$

After foil 2 the intensity is

$$I_2 = I_o(1 + G_f)^2 \quad (34)$$

In the same way, after foil N the intensity is:

$$I_N = I_o(1 + G_f)^N \quad (35)$$

The total gain G after N foils is given as:

$$G = (1 + G_f)^N - 1 \quad (36)$$

The radiation output power P_N is given from (35) as:

$$P_N = P_o(1 + G_f)^N, \quad (37)$$

where P_o is the incident radiation power.

To be useful, the laser gain G must be greater than other losses such as absorption in the target. The choice of material will be such that absorption is minimized. As an example, using typical values from [15], the laser gain for ħ ω = 2 KeV using 200 Beryllium foils of thickness 4.1 micrometer with plasma frequency $\omega_p = 25\text{eV}$, with $\varepsilon_2 = 0.999844$, $\varepsilon_1 = 1$, the peak current density, J = 10^8 A /mm^2, E = 5GeV, (by (26) and (36)) is 5.1%

For incident radiation power P_o = 1W from the thin foils stack target, the output power for the 200 foils in the laser medium is 1.05W. To achieve higher gain we increase the number of targets.

D Coherent Gain

We propose in this section the existence of coherent gain when the electron beams are pre-bunched such that the bunch length is comparable or less than the wavelength of the radiation. This is akin to the phenomenon of superradiance (coherent spontaneous emission) and stimulated superradiance [18,19] which occur when the electron bunch length is shorter than the radiation wavelength. In the case of superradiance the spontaneous emission is enhanced by a factor of $N_eF(\lambda)$ where λ is the wavelength, N_e is the number of electrons in the bunch. F(λ) is the bunch form factor which is given by the Fourier transform of the distribution function of the electrons in the bunch. The basic explanation for the existence of superradiance is that the radiation field wave packets from the electrons in the bunch add up constructively so that the sum of the fields is proportional to number of electrons in the bunch and thus the radiation intensity will be proportional to the square of the number of electrons in the bunch. From quantum theoretical point of view we may imagine that superradiance occurs because the electrons interacts with vacuum photon fields as a unit so that the probability amplitude of emission is proportional to the number of charges in the bunches and thus the probability of emission will be proportional to the square of the number of electrons in the bunch. From this point view, the electron bunch will also interact as a unit with an incident radiation leading to stimulated superradiance which has been confirmed experimentally [19]. Stimulated superradiance is more intense than superradiance and is proportional to square of the number of electrons in the bunch and the intensity of incident stimulating radiation. Also the cooperative interaction of the electrons in the bunch with incident radiation will also lead to super absorption which will be proportional to the square of the number of electrons in the bunch in addition to the radiation intensity. Therefore in our laser configuration, if we have an electron beam consisting of bunches of length shorter than the wavelength of the radiation, we expect that both the stimulated radiation intensity and stimulated absorption given by (8) and (11) respectively will be enhanced by a factor of $N_eF(\lambda)$. From (12) and (25), the coherent gain coefficient per foil, g_c is given by:

$$g_c = (1 + N_eF(\lambda))G_f, \quad (38)$$

where G_f is the gain coefficient as given by (25).

We can see that the gain is greatly enhanced due to coherence. As an example, the laser gain assuming coherence using 15 Beryllium foils of thickness 4.1 micrometer for ħ ω = 2 keV, with $\varepsilon_2 = 0.999844$, $\varepsilon_1 = 1$, the peak current density, J = 10^4 A /mm^2, E = 5GeV, Ne = 10^6, F(λ) = 0.9 ,(by (26) and (38)) is 40%. For incident radiation power P_o = 1W from the thin foils stack target, the output power for the 15 foils in the laser medium is 1.4W. To achieve higher gain we increase the number of targets. By increasing the number of foils in the laser to 150 the output power becomes 28W.

VI. CONCLUSION

In summary, we have analyzed a model of a transition radiation laser based on stimulated processes occurring at the boundary between two dielectric media as a result of interaction between incident monochromatic radiation and relativistic electron beams crossing the interface. The operation of the laser requires successive overlap of the electron bunches and the radiation at the boundaries of the dielectric media. This overlap requires a matching condition. The condition for this matching has been derived. Since the laser fields develop from the spontaneous emissions, a crystal monochrometer is fixed to select a chosen wavelength.

Phenomenological quantum electrodynamics is applied to derive analytical expressions for stimulated emission and absorption probabilities. Analogs of Einstein's coefficients for the transition processes have also been derived and discussed. It is shown that stimulated emission is greater than absorption and thus there is a possibility for amplification of radiation.

The derived analytical expressions are used for the calculation of the gain. The gain is greatly enhanced in the case of coherence. But it may be difficult to have an electron bunches shorter than EUV and X-ray wavelength with significant charge per bunch which is necessary to have coherent gain.

The gain can also be increased by increasing the beam current and subtly finding ways to suppress stimulated absorption.

This will constitute the subject of our next study in finding ways of improving on the gain. In our next paper, we will also consider the effect of multiple scattering of the electrons as the number of targets increases particularly for low energy electrons.

The laser system has the potential of being developed into a compact laser source particularly in the frequency range where electron bunches shorter than the wavelength are readily available. It can also serve as an energy recovery system attached to a conventional FEL. The electron beams exiting the wiggler fields of the FEL, being bunched to some degree, can serve as a source of bunched electron beam needed to achieve the coherent gain.

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